Canonical quantization of nonlinear many body systems

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Abstract

We study the quantization of a classical system of interacting particles obeying a recently proposed kinetic interaction principle (KIP) [G. Kaniadakis, Physica A **296**, 405 (2001)]. The KIP fixes the expression of the Fokker-Planck equation describing the kinetic evolution of the system and imposes the form of its entropy. In the framework of canonical quantization, we introduce a class of nonlinear Schrödinger equations (NSEs) with complex nonlinearities, describing, in the mean field approximation, a system of collectively interacting particles whose underlying kinetics is governed by the KIP. We derive the Ehrenfest relations and discuss the main constants of motion arising in this model. By means of a nonlinear gauge transformation of third kind it is shown that in the case of constant diffusion and linear drift the class of NSEs obeying the KIP is gauge-equivalent to another class of NSEs containing purely real nonlinearities depending only on the field $\rho = |\psi|^2$.

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I. INTRODUCTION

A wide class of diffusive processes in nature, known as normal diffusion, are successfully described by the linear Fokker-Planck equation. Its relation to Boltzmann-Gibbs entropy (BG-entropy) in the framework of the irreversible thermodynamics is well established [1, 2, 3].

However, nonlinear Fokker-Planck equations (NFPEs) [4, 5, 6, 7, 8] and their connection in the field of the generalized thermodynamics [9, 10, 11] is nowadays an intense research area. In particular, many physical phenomena, in presence of memory effects, nonlocal effects, long-range effects or, more in general, nonlinear effects, are well understood with the help of NFPEs.

To cite a few, let us recall the problem of diffusion in polymers [12], on liquid surfaces [13], in Lévy flights [14] and enhanced diffusion in active intracellular transport [15]. Many anomalous diffusion systems have a quantum nature, like for instance charge transport in anomalous solids [16], subrecoil laser cooling [17] and the aging effect in quantum dissipative systems [18].

A still open question concerns the dynamics underlying the nonlinear kinetics governing the above anomalous systems. Langevin-like, Fokker-Planck-like or Boltzmann-like equations have been used by different authors to generate nonlinear terms in the Schrödinger equation with the aim of describing, in the mean field approximation, the many quantum particle interactions [19, 20, 21, 22].

It is now widely recognized that the presence of a nonlinear drift term as well as the presence of a diffusive term in a quantum particle current originates complex nonlinearities in the evolution equation for the ψ-wave function.

Different examples are known in literature of nonlinear Schrödinger equations (NSEs) originating from the study of the kinetics governing the many-body quantum system. For instance, the Doebner-Goldin family equations [23] have been introduced from topological considerations as the most general class of Schrödinger equations compatible with the linear Fokker-Planck equation. In Ref. [24] the authors introduced a NSE starting from a generalized exclusion-inclusion principle (EIP) in order to describe systems of quantum particles with different statistics interpolating with continuity between the Bose-Einstein and the Fermi-Dirac ones. In Ref. [25], in the stochastic quantization framework, starting from

the most general nonlinear kinetics containing a nonlinear drift term and compatible with a linear diffusion term, a class of NSEs with a complex nonlinearity was obtained.

Recently, a kinetic interaction principle (KIP) has been proposed [26] to define a special collective interaction among the N-identical particles of a classical system. On the one hand, the KIP imposes the form of the generalized entropy associated with the system, while on the other hand it governs the evolution of the system toward equilibrium by fixing the expression of the nonlinear current of particles in the NFPE, thus governing the kinetics underlying the system.

The link between the generalized entropic functional and the corresponding NFPE can also be obtained starting from a maximum entropic production principle. In Refs. [6, 7], taking into account a variational principle maximizing the dissipation rate of a generalized free energy, the authors obtained a NFPE in the Smoluchowski limit. The same NFPE was obtained in Ref. [8] from a stochastic process described by a generalized Langevin equation where the strength of the noise is assumed to depend on the density of the particle.

In the present paper we perform the quantization of a classical system obeying KIP, where the statistical information is supplied by a very general entropy.

Up to today, different methods have been proposed for the microscopic description of systems. Schrödinger's wave mechanics, Heisenberg's matrix mechanics or Feynman's path-integral mechanics are some of the many. Another approach is given by the hydrodynamic theory of quantum mechanics originally owing to Madelung [27] and de Broglie [28] and successively reconsidered by Bohm [29] in connection with his theory of hidden variables.

In the hydrodynamic formulation of quantum mechanics, the complex linear Schrödinger equation is replaced by two real nonlinear differential equations for two independent fields: the probability density and its velocity field. Basically, such equations are formally similar to the equations of continuity and the Euler equation of ordinary hydrodynamics.

This formalism is fruitful, as in the present situation, when the expression of the quantum continuity equation is inherited from the one describing the kinetics of the ancestor classical system. However, for a complete quantum mechanical description, besides the continuity equation, we need to know if and how we should generalize the Euler equation that describes the dynamics of the system. In this paper, in order to fix the nonlinear terms in the Euler equation, we require that the whole model be formulated in the canonical formalism.

We obtain a class of NSEs with complex nonlinearity describing a quantum system of inter-

acting particles obeying the KIP in the mean field approximation. We study the case of a quantum system undergoing a constant diffusion process. The generalization to the case of a nonconstant diffusive process is also presented at the end of the paper. It is shown that the form of the entropy of the ancestor classical system fixes the nonlinearity appearing in the evolution equation. By means of a recently proposed nonlinear gauge transformation [23, 30, 31] this family of evolution equations is transformed into another one describing a nondiffusive process. In particular, when the kinetics of the system is governed by a linear drift term, the new family of NSEs contains a purely real nonlinearity depending only on the density of particles $\rho = |\psi|^2$.
As working examples we present the quantization of some classical systems described by entropies already known in the literature: BG-entropy, Tsallis-entropy [32], Kaniadakis-entropy [26] and the interpolating quantum statistics entropy [33].

The plan of the paper is the following. In Section II we recall the relation between a given generalized entropy and the associated NFPE describing the kinetic evolution of the classical system in the nonequilibrium thermodynamic framework. This kinetic equation is justified on the ground of KIP. In Section III, firstly first present an overall summing up of the hydrodynamic formulation of the linear Schrödinger equation, then we generalize the method to quantize the classical system obeying EIP. The Hamiltonian formulation of this model is presented and a family of NSEs with complex nonlinearity is obtained. In Section IV we study the Ehrenfest relations and discuss the conserved mean quantities. In Section V, the nonlinear gauge transformation is introduced. Some relevant examples are presented in Section VI. The final Section VII present comments and conclusions. In Appendix A we give the derivation of the Ehrenfest relations while in Appendix B we briefly discuss the generalization of the model for a quantum system whose kinetics undergoes a nonconstant diffusive process.

II. NONLINEAR FOKKER-PLANCK EQUATION

Our starting point, according to nonlinear kinetics, is to relate the production of the entropy of a classical system to a Fokker-Planck equation. This can be accomplished by following the classical approach to diffusion [1, 2].
We start by assuming a very general trace-form expression for the entropy (throughout this

paper, we use units with the Boltzmann constant $k_{\rm B}$ set equal to unity)

$$S(\rho) = -\int d\boldsymbol{x} \int d\rho \, \ln \kappa(\rho) \ , \tag{2.1}$$

where $\kappa(\rho)$ is an arbitrary functional of the density particles field $\rho = \rho(t, \boldsymbol{x})$, with $\boldsymbol{x} \equiv (x_1, \cdots, x_n)$ a point in the n-dimensional space.

The constraints

$$\int \rho \, d\boldsymbol{x} = 1 \ , \tag{2.2}$$

on the normalization and

$$\int \mathcal{E}(\boldsymbol{x}) \, \rho \, d\boldsymbol{x} = E \ , \tag{2.3}$$

total energy of the system, with $\mathcal{E}(\boldsymbol{x}) = \boldsymbol{p}^2/2\,m + V(\boldsymbol{x})$ the energy for each particle, are accounted for by introducing the constrained entropic functional

$$\mathcal{S}(\rho) = -\int d\boldsymbol{x} \int d\rho \, \ln \kappa(\rho) - \beta \int \mathcal{E}(\boldsymbol{x}) \, \rho \, d\boldsymbol{x} - \beta' \int \rho \, d\boldsymbol{x} \ . \tag{2.4}$$

The two constants β and β' are the Lagrange multipliers associated with constraints (2.2) and (2.3).

Quite generally, the evolution of the field ρ in the configuration space is governed by the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{J} = 0 \ , \tag{2.5}$$

with $\boldsymbol{\nabla} \equiv (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, and assures the conservation of the constraint (2.2) in time. We assume a nonlinear relation between the current $\boldsymbol{J}$ and the constrained thermodynamic force

$$\boldsymbol{\mathcal{F}}(\rho) = \boldsymbol{\nabla} \left(\frac{\delta \mathcal{S}}{\delta \rho} \right) \ , \tag{2.6}$$

by posing

$$\boldsymbol{J} = D \, \gamma(\rho) \, \boldsymbol{\mathcal{F}}(\rho) \ , \tag{2.7}$$

with D the diffusion coefficient and $\gamma(\rho)$ still an arbitrary functional of ρ.

Putting Eq. (2.7) in Eq. (2.5), and taking into account the expression of $\mathcal{S}$ given in Eq. (2.4) we obtain the following continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \Big\{ - D \, \gamma(\rho) \boldsymbol{\nabla} \Big[\beta \, \mathcal{E}(\boldsymbol{x}) + \beta' + \ln \kappa(\rho) \Big] \Big\} = 0 \ . \tag{2.8}$$

Introducing drift velocity

$$\boldsymbol{u}_{\rm drift} = -D \, \beta \, \boldsymbol{\nabla} \, \mathcal{E}(\boldsymbol{x}) \ , \tag{2.9}$$

Eq. (2.8) takes the form of a NFPE for the field ρ

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\Big[\boldsymbol{u}_{\rm drift}\,\gamma(\rho)-D\,f(\rho)\,\boldsymbol{\nabla}\,\rho\Big]=0\ , \tag{2.10}$$

where

$$f(\rho)=\gamma(\rho)\,\frac{\partial\,\ln\,\kappa(\rho)}{\partial\,\rho}\ . \tag{2.11}$$

Total current $\boldsymbol{J}=\boldsymbol{J}_{\rm drift}+\boldsymbol{J}_{\rm diff}$ is the sum of a nonlinear drift current $\boldsymbol{J}_{\rm drift}=\boldsymbol{u}_{\rm drift}\,\gamma(\rho)$, and a nonlinear diffusion current $\boldsymbol{J}_{\rm diff}=-D\,f(\rho)\,\boldsymbol{\nabla}\,\rho$, different from Fick's standard one $\boldsymbol{J}_{\rm Fick}=-D\,\boldsymbol{\nabla}\,\rho$, which is recovered by posing $\gamma(\rho)=\kappa(\rho)=\rho$.

Eq. (2.10) describes a class of nonlinear diffusive processes varying the functionals $\gamma(\rho)$ and $\kappa(\rho)$.

We observe that for any given entropy (2.1) an infinity of associated NFPEs exists, one for any choice of $\gamma(\rho)$.

In Refs. [6, 7], starting from a variational principle which maximizes the dissipation rate of a generalized free energy functional, substantially equivalent to Eq. (2.4), a NFPE in the position space as in Eq. (2.10) has been obtained. The same NFPE (2.10) was also obtained in Ref. [8], starting from a stochastic process described by a generalized Langevin equation, where the strength of the noise is assumed to depend on the density of the particle. The nonlinear current, as in Eq. (2.7), is given by the gradient of the functional derivative of a generalized free energy equivalent to Eq. (2.4).

In Ref. [4] the problem of the NFPE derived from generalized linear nonequilibrium thermodynamics was also discussed at length.

At equilibrium, the particle current must vanish, and from Eq. (2.6) it follows

$$\ln\kappa(\rho_{\rm eq})+\beta\,\mathcal{E}(\boldsymbol{x})+\beta'=0\ , \tag{2.12}$$

where, without loss of generality, we posed the integration constant equal to zero (otherwise it can be included in the Lagrange multiply β').

We obtain the equilibrium distribution of the system

$$\rho_{\rm eq}=\kappa^{-1}\Big(\exp\left(-\beta\,\mathcal{E}(\boldsymbol{x})-\beta'\right)\Big)\ . \tag{2.13}$$

In particular, with the choice $\kappa(\rho)=e\,\rho$, Eq. (2.1) reduces to standard BG-entropy and Eq. (2.13) gives the well-known Gibbs-distribution.

Let us now justify Eq. (2.10) starting from the kinetic approach introduced in [26] through the KIP. In accordance with the arguments presented in Ref. [26], we consider the following classical Markovian process

$$\frac{\partial \rho}{\partial t} = \int \left[\pi(t, \boldsymbol{y} \to \boldsymbol{x}) - \pi(t, \boldsymbol{x} \to \boldsymbol{y})\right] \, d\boldsymbol{y} \ , \tag{2.14}$$

describing the kinetics of a system of N-identical interacting particles.
For transition probability $\pi(t, \boldsymbol{x} \to \boldsymbol{y})$ we assume a suitable expression in terms of the populations of the initial site $\boldsymbol{x}$ and the final site $\boldsymbol{y}$.
According to KIP we pose

$$\pi(t, \boldsymbol{x} \to \boldsymbol{y}) = r(t, \, \boldsymbol{x}, \, \boldsymbol{x} - \boldsymbol{y}) \, \gamma(\rho, \, \rho') \ , \tag{2.15}$$

where $\rho \equiv \rho(t, \, \boldsymbol{x})$ and $\rho' \equiv \rho(t, \, \boldsymbol{y})$ are the particle density functions in the starting site $\boldsymbol{x}$ and in the arrival site $\boldsymbol{y}$ respectively, whereas $r(t, \, \boldsymbol{x}, \, \boldsymbol{x} - \boldsymbol{y})$ is the transition rate which depends only on the starting $\boldsymbol{x}$ and arrival $\boldsymbol{y}$ sites, during particle transition $\boldsymbol{x} \to \boldsymbol{y}$.
The functional $\gamma(\rho, \, \rho')$ can be factorized in

$$\gamma(\rho, \, \rho') = a(\rho) \, b(\rho') \, c(\rho, \, \rho') \ . \tag{2.16}$$

The first factor $a(\rho)$ is a functional of the particle population ρ of the starting site and satisfies the boundary condition $a(0) = 0$, since if the starting site is empty transition probability is equal to zero. The second factor $b(\rho')$ is a functional of the particle population ρ' at the arrival site, and satisfies the condition $b(0) = 1$, because the transition probability does not depend on the arrival site if particles are absent there. Finally, the third factor $c(\rho, \, \rho')$ takes into account that the populations of the two sites can eventually affect the transition collectively and symmetrically.
The expression of the functional $b(\rho')$ plays a very important role in the particle kinetics because it can stimulate or inhibit the transition $\boldsymbol{x} \to \boldsymbol{y}$, allowing, in this way, interactions originating from collective effects.
With the assumptions made in Eqs. (2.15) and (2.16) for transition probability, according to the Kramers-Moyal expansion and assuming the first neighbor approximation, we can expand up to the second order the quantities $r(t, \, \boldsymbol{y}, \, \boldsymbol{y} - \boldsymbol{x}) \, \gamma(\rho(t, \, \boldsymbol{y}), \, \rho(t, \, \boldsymbol{x}))$ and $\gamma(\rho(t, \, \boldsymbol{x}), \, \rho(t, \, \boldsymbol{y}))$ in Taylor series of $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{u}$ and $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{u}$, respectively, in an interval around $\boldsymbol{x}$, for $\boldsymbol{u} \ll \boldsymbol{x}$.

We obtain

$$
\begin{aligned}
& r(t,\, \boldsymbol{x}+\boldsymbol{u},\, \boldsymbol{u})\, \gamma\Big(\rho(t,\, \boldsymbol{x}+\boldsymbol{u}),\, \rho(t,\, \boldsymbol{x})\Big) \\
= & \left\{ r(t,\, \boldsymbol{y},\, \boldsymbol{u})\, \gamma\Big(\rho(t,\, \boldsymbol{y}),\, \rho(t,\, \boldsymbol{x})\Big) + \frac{\partial}{\partial\, y_i}\Big[r(t,\, \boldsymbol{y},\, \boldsymbol{u})\, \gamma\Big(\rho(t,\, \boldsymbol{y}),\, \rho(t,\, \boldsymbol{x})\Big)\Big]\, u_i \right. \\
& \left. + \frac{1}{2}\, \frac{\partial^2}{\partial\, y_i\, \partial\, y_j}\Big[r(t,\, \boldsymbol{y},\, \boldsymbol{u})\, \gamma\Big(\rho(t,\, \boldsymbol{y}),\, \rho(t,\, \boldsymbol{x})\Big)\Big]\, u_i\, u_j \right\}_{\boldsymbol{y}\to\boldsymbol{x}} \quad ,
\end{aligned} \tag{2.17}
$$

and

$$
\begin{aligned}
\gamma\Big(\rho(t,\, \boldsymbol{x}),\, \rho(t,\, \boldsymbol{x}-\boldsymbol{u})\Big) \quad = & \left\{ \gamma\Big(\rho(t,\, \boldsymbol{x}),\, \rho(t,\, \boldsymbol{y})\Big) - \frac{\partial}{\partial\, y_i} \gamma\Big(\rho(t,\, \boldsymbol{x}),\, \rho(t,\, \boldsymbol{y})\Big)\, u_i \right. \\
& \left. + \frac{1}{2}\, \frac{\partial^2}{\partial\, y_i\, \partial\, y_j}\, \gamma\Big(\rho(t,\, \boldsymbol{x}),\, \rho(t,\, \boldsymbol{y})\Big)\, u_i\, u_j \right\}_{\boldsymbol{y}\to\boldsymbol{x}} \quad .
\end{aligned} \tag{2.18}
$$

Using Eqs. (2.17) and (2.18) in Eq. (2.15), from Eq. (2.14) it follows

$$
\frac{\partial\, \rho}{\partial\, t} = \frac{\partial}{\partial x_i}\left[\left(\zeta_i + \frac{\partial\, \zeta_{ij}}{\partial x_j}\right)\, \gamma(\rho) + \zeta_{ij}\, \gamma(\rho)\, \frac{\partial}{\partial\, x_j}\, \ln \kappa(\rho)\right] \ , \tag{2.19}
$$

with $i = 1,\, \cdots,\, n$ and summation over repeated indices is assumed.
In Eq. (2.19)

$$
\gamma(\rho) \equiv \gamma(\rho,\, \rho')\Big|_{\rho=\rho'} \ , \tag{2.20}
$$

and

$$
\kappa(\rho) = \frac{a(\rho)}{b(\rho)} \ , \tag{2.21}
$$

while the coefficients ζ_i and ζ_{ij} are given by

$$
\zeta_i = \int r(t,\, \boldsymbol{y},\, \boldsymbol{u})\, u_i\, d\boldsymbol{u} \ , \tag{2.22}
$$

$$
\zeta_{ij} = \frac{1}{2} \int r(t,\, \boldsymbol{y},\, \boldsymbol{u})\, u_i\, u_j\, d\boldsymbol{u} \ . \tag{2.23}
$$

Defining $(u_i)_{\rm drift} = -\zeta_i - \partial\, \zeta_{ij}/\partial\, x_j$, the i-th component of $\boldsymbol{u}_{\rm drift}$, and assuming the independence of motion in different directions of the isotropic configuration space we can pose $\zeta_{ij} = D\, \delta_{ij}$. It is easy to see that Eq. (2.19) reduces to Eq. (2.10).

In conclusion we observe that Eq. (2.10) is a NFPE in the Smoluchowski limit since it describes a kinetic process in the position space rather than in the phase space. This is a suitable form for the quantum treatment of the following sections. The passage from the NFPE in the phase space to the NFPE in the position space was rigorously elaborated in Ref. [34] in the limit of strong friction, by means of a Chapman-Enskog-like expansion.

III. CANONICAL QUANTIZATION

A. Quantization in the hydrodynamic representation

In the hydrodynamic representation, the quantum mechanics formulation, can readily be obtained from the standard Schrödinger equation

$$i\hbar\,\frac{\partial\,\psi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\,\Delta\,\psi+V(\boldsymbol{x})\,\psi\ , \tag{3.1}$$

where $V(\boldsymbol{x})$ is a real external potential. The complex field $\psi\equiv\psi(t,\,\boldsymbol{x})$ describing the quantum system is related to the hydrodynamic fields $\rho(t,\,\boldsymbol{x})$ and $\Sigma(t,\,\boldsymbol{x})$ through polar decomposition [27, 29]

$$\psi(t,\,\boldsymbol{x})=\rho^{1/2}(t,\,\boldsymbol{x})\,\exp\left(\frac{i}{\hbar}\,\Sigma(t,\,\boldsymbol{x})\right)\ . \tag{3.2}$$

Eq. (3.1) is separated into a couple of real equations

$$m\,\frac{\partial\,\widehat{\boldsymbol{v}}}{\partial\,t}+m\,\left(\widehat{\boldsymbol{v}}\cdot\boldsymbol{\nabla}\right)\,\widehat{\boldsymbol{v}}=\boldsymbol{\nabla}\,\left(\frac{\hbar^2}{2\,m}\,\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}}-V(\boldsymbol{x})\right)\ , \tag{3.3}$$

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}_{_0}=0\ , \tag{3.4}$$

where quantum velocity $\widehat{\boldsymbol{v}}$, which in the linear case coincides with quantum drift velocity $\widehat{\boldsymbol{u}}_{\rm drift}$, is related to the phase $\Sigma(t,\,\boldsymbol{x})$ through

$$m\,\widehat{\boldsymbol{v}}=\boldsymbol{\nabla}\,\Sigma(t,\,\boldsymbol{x})\ , \tag{3.5}$$

and

$$\boldsymbol{j}_{_0}=\rho\,\widehat{\boldsymbol{v}}\ , \tag{3.6}$$

is the same relationship between current and velocity of the standard hydrodynamic theory. We remark that the quantum current (3.6) contains only a linear drift term.
According to the orthodox interpretation of quantum mechanics the quantity $\rho(t,\,\boldsymbol{x})=|\psi(t,\,\boldsymbol{x})|^2$ represents the position probability density of the system normalized as $\int\rho(t,\,\boldsymbol{x})\,d\boldsymbol{x}=1$.

Eqs. (3.3)-(3.6) form the basis of the hydrodynamic formulation which consists of a quasi classical approach to quantum mechanics. In this picture the evolution of the system can be interpreted in terms of a flowing fluid with density $\rho(t,\,\boldsymbol{x})$ associated with a local velocity field $\widehat{\boldsymbol{v}}(t,\,\boldsymbol{x})$. The dynamics of such fluid is described by the Euler equation (3.3)

and is governed by forces arising not only from the external field $\boldsymbol{F}_{\rm ext}(\boldsymbol{x}) = -\boldsymbol{\nabla}\, V(\boldsymbol{x})$ but also from an additional potential $U_q = -(\hbar^2/2\,m)\,\Delta\sqrt{\rho}/\sqrt{\rho}$ known as the quantum potential [29]. Remarkably, the expectation value for the quantum force vanishes at all times, i.e. $\langle -\boldsymbol{\nabla}\, U_q\rangle = 0$. Finally, the continuity equation (3.4) assures the conservation of the normalization of wave function ψ during the evolution of the system.

Let us remark that the quantum fluid has a very special property. Because $\Sigma(t,\, \boldsymbol{x})$ is a potential field for the quantum velocity, the quantum fluid is irrotational. As a consequence, in the linear Schrödinger theory, a non vanishing vorticity $\boldsymbol{\omega}$, defined by

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \widehat{\boldsymbol{v}} \ , \tag{3.7}$$

is possible only at the nodal region where neither $\Sigma(t,\, \boldsymbol{x})$ nor $\boldsymbol{\nabla}\,\Sigma(t,\, \boldsymbol{x})$ are well defined. At such a point $\boldsymbol{\nabla} \times \boldsymbol{\nabla}\,\Sigma(t,\, \boldsymbol{x})$ does not vanish in general, thus leading to the appearance of point-like vortices.

Finally, putting Eq. (3.5) into Eq. (3.3) we obtain

$$\frac{\partial\,\Sigma}{\partial\, t} + \frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m} - \frac{\hbar^2}{2\,m}\,\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} + V(\boldsymbol{x}) = 0 \ . \tag{3.8}$$

This equation, in the classical limit $\hbar \to 0$, reduces to the Hamilton-Jacobi equation for the function Σ.

Eqs. (3.4) and Eq. (3.8) can be obtained by means of the Hamiltonian equations

$$\frac{\partial\,\Sigma}{\partial\, t} = -\frac{\delta\, H}{\delta\,\rho} \ , \tag{3.9}$$

$$\frac{\partial\,\rho}{\partial\, t} = \frac{\delta\, H}{\delta\,\Sigma} \ , \tag{3.10}$$

where the Hamiltonian

$$H = \int \mathcal{H}(\rho,\, \Sigma)\, d\boldsymbol{x} \ , \tag{3.11}$$

with

$$\mathcal{H}(\rho,\, \Sigma) = \frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}\,\rho + \frac{\hbar^2}{8\,m}\,\frac{(\boldsymbol{\nabla}\,\rho)^2}{\rho} + V(\boldsymbol{x})\,\rho \ . \tag{3.12}$$

represents the total energy of the quantum system.

B. The many-body quantum system

Let us now generalize the method described above by replacing the linear continuity equation Eq. (3.4) with the more general one obtained in analogy with the continuity

equation (2.10) describing the kinetics of a classical system obeying KIP. In the following we assume that the quantum system undergoes a constant diffusion process with $D=const.$

We begin by introducing the wave function $\psi\equiv\psi(t,\,\boldsymbol{x})$ describing, in the mean field approximation, a system of quantum interacting particles. We postulate that the following NSE describes the evolution equation of the system

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\,\Delta\,\psi+\Lambda(\psi^*,\,\psi)\,\psi+V(\boldsymbol{x})\,\psi\ , \tag{3.13}$$

where $\Lambda(\psi^*,\,\psi)=W(\psi^*,\,\psi)+i\,\mathcal{W}(\psi^*,\,\psi)$ is a complex nonlinearity, with $W(\psi^*,\,\psi)$ and $\mathcal{W}(\psi^*,\,\psi)$ the real and the imaginary part, respectively.

Using polar decomposition (3.2), Eq. (3.13) is separated into a couple of real nonlinear evolution equations for phase and amplitude

$$\frac{\partial\,\Sigma}{\partial\,t}+\frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}+U_q+W(\rho,\,\Sigma)+V(\boldsymbol{x})=0\ , \tag{3.14}$$

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\boldsymbol{j}_{_0}-\frac{2}{\hbar}\,\rho\,\mathcal{W}(\rho,\,\Sigma)=0\ . \tag{3.15}$$

We require that both Eqs. (3.14) and (3.15) can be obtained through the Hamilton equations (3.9)-(3.10) and, to accommodate nonlinearities $W(\rho,\,\Sigma)$ and $\mathcal{W}(\rho,\,\Sigma)$, we introduce in the Hamiltonian density $\mathcal{H}$ an additional real nonlinear potential $U(\rho,\,\Sigma)$ which describes the collective interaction between the particles belonging to the system

$$\mathcal{H}(\rho,\,\Sigma)=\frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}\,\rho+\frac{\hbar^2}{8\,m}\,\frac{(\boldsymbol{\nabla}\,\rho)^2}{\rho}+U(\rho,\,\Sigma)+V(\boldsymbol{x})\,\rho\ . \tag{3.16}$$

By means of Eqs. (3.9) and (3.10) it follows that the nonlinear functionals $W(\rho,\,\Sigma)$ and $\mathcal{W}(\rho,\,\Sigma)$ are related to the nonlinear potential $U(\rho,\,\Sigma)$ as

$$W(\rho,\,\Sigma)=\frac{\delta}{\delta\,\rho}\int U(\rho,\,\Sigma)\,d\boldsymbol{x}\ , \tag{3.17}$$

$$\mathcal{W}(\rho,\,\Sigma)=\frac{\hbar}{2\,\rho}\,\frac{\delta}{\delta\,\Sigma}\int U(\rho,\,\Sigma)\,d\boldsymbol{x}\ . \tag{3.18}$$

We assume that the quantum fluid satisfies a continuity equation formally equal to the classical one described by the NFPE (2.10). By matching Eq. (3.15) with Eq. (2.10) we obtain the expression $\mathcal{W}$ and, accounting for Eq. (3.18), we have the nonlinear potential $U(\rho,\,\Sigma)$. Finally, the nonlinearity $W(\rho,\,\Sigma)$, which follows from Eq. (3.17), together with the quantum potential U_q and the external potential $V(\boldsymbol{x})$, describes the dynamic behavior of the quantum fluid according to Eq. (3.14).

We observe that if the following equation holds

$$\frac{\delta}{\delta\,\Sigma}\int U(\rho,\,\Sigma)\,d\boldsymbol{x}=\boldsymbol{\nabla}\cdot\boldsymbol{F}(\rho,\,\Sigma)\ , \tag{3.19}$$

with $\boldsymbol{F}(\rho,\,\Sigma)$ an arbitrary functional, taking into account Eq. (3.18), Eq. (3.15) becomes

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\Big[\boldsymbol{j}_{_0}-\boldsymbol{F}(\rho,\,\Sigma)\Big]=0\ . \tag{3.20}$$

Eq. (3.19) is fulfilled if functional $U(\rho,\,\Sigma)$ depends on phase Σ only through its spatial derivatives [30].

Introducing the quantum drift velocity

$$\widehat{\boldsymbol{u}}_{\rm drift}=\frac{\boldsymbol{\nabla}\,\Sigma}{m}\ , \tag{3.21}$$

which in the linear case coincides with the quantum velocity $\widehat{\boldsymbol{v}}$ given in Eq. (3.5), and by comparing Eq. (3.20) with Eq. (2.10) we have

$$\boldsymbol{F}(\rho,\,\Sigma)=\frac{\boldsymbol{\nabla}\,\Sigma}{m}\Big[\rho-\gamma(\rho)\Big]+D\,f(\rho)\ \boldsymbol{\nabla}\,\rho\ . \tag{3.22}$$

By integrating Eq. (3.18), the nonlinear potential assumes the expression

$$U(\rho,\,\Sigma)=\frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}\Big[\gamma(\rho)-\rho\Big]-D\,f(\rho)\,\boldsymbol{\nabla}\rho\cdot\boldsymbol{\nabla}\Sigma+\widetilde{U}(\rho)\ , \tag{3.23}$$

where $\widetilde{U}(\rho)$ is an arbitrary real potential depending only on field ρ. Eqs. (3.9) and (3.10) give the following coupled nonlinear evolution equations

$$\frac{\partial\,\Sigma}{\partial\,t}+\frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}\,\frac{\partial\,\gamma(\rho)}{\partial\,\rho}-\frac{\hbar^2}{2\,m}\,\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}}+m\,D\,f(\rho)\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}_{_0}}{\rho}\right)+G(\rho)+V(\boldsymbol{x})=0\ , \tag{3.24}$$

$$\frac{\partial\,\rho}{\partial\,t}+\boldsymbol{\nabla}\cdot\left[\frac{\boldsymbol{\nabla}\,\Sigma}{m}\,\gamma(\rho)-D\,f(\rho)\,\boldsymbol{\nabla}\,\rho\right]=0\ , \tag{3.25}$$

where $G(\rho)=\delta\,\int\widetilde{U}(\rho)\,d\boldsymbol{x}/\delta\,\rho$.

In the classical limit $\hbar\to 0$ Eq. (3.24) becomes a nonlinear Hamilton-Jacobi equation for function Σ. It differs from the classical one owing to the presence of the nonlinear term which functionally depends on both ρ and Σ. We recall that such a nonlinearity was introduced consistently with the requirement of a final canonical formulation of the theory.

We stress once again that in the approach described in this paper, we start from a nonlinear

generalization of the continuity equation that gives us only information on the kinetics. This equation is not enough to completely determine the time evolution of the quantum system. As a consequence, we have ample freedom in the definition of nonlinear potential $U(\rho,\,\Sigma)$. Such freedom is reflected in the arbitrary functional $\widetilde{U}(\rho)$ which cannot be fixed only on the basis of the kinetic equation. There are many possible dynamic behaviors, one for any choice of $\widetilde{U}(\rho)$, compatible with the same kinetics. The nonlinear potential $\widetilde{U}(\rho)$ can be used to describe other possible interactions among the many particles of the system that have an origin different from the one introduced by the kinetic equation (3.25).

Actually, Eq. (3.25) is a quantum continuity equation for field ρ with a nonlinear quantum current given by

$$\boldsymbol{j}=\frac{\boldsymbol{\nabla}\,\Sigma}{m}\,\gamma(\rho)-D\,f(\rho)\,\boldsymbol{\nabla}\,\rho\ . \tag{3.26}$$

We observe that, differently from the hydrodynamic formulation of the linear quantum mechanics, where the Bohm-Madelung fluid is irrotational, in nonlinear quantum theory the situation can be very different. In fact, by defining quantum velocity through Eq. (3.6), from Eq. (3.26) we have

$$m\,\widehat{\boldsymbol{v}}=\frac{\gamma(\rho)}{\rho}\,\boldsymbol{\nabla}\left[\Sigma-m\,D\,\ln\kappa(\rho)\right]\ , \tag{3.27}$$

which states the relationship between quantum velocity $\widehat{\boldsymbol{v}}$ and quantum drift velocity $\widehat{\boldsymbol{u}}_{\rm drift}$ for the nonlinear case.

Expression (3.27) can be justified in terms of Clebsh potentials. In fact, as is well known, a nonvanishing vorticity can be accounted for in the Schrödinger theory by introducing three potentials μ, ν and λ related to quantum velocity through the relation

$$m\,\widehat{\boldsymbol{v}}=\boldsymbol{\nabla}\,\mu+\nu\,\boldsymbol{\nabla}\,\lambda\ . \tag{3.28}$$

Vorticity $\boldsymbol{\omega}$ assumes a nonvanishing expression given by

$$\boldsymbol{\omega}=\frac{1}{m}\boldsymbol{\nabla}\,\nu\times\boldsymbol{\nabla}\,\lambda\ . \tag{3.29}$$

By comparing Eq. (3.28) with Eq. (3.27) we readily obtain $\mu=const$, $\nu=\gamma(\rho)/\rho$ and $\lambda=\Sigma-m\,D\,\ln\kappa(\rho)$, respectively, and Eq. (3.29) becomes

$$\boldsymbol{\omega}=\frac{1}{m}\boldsymbol{\nabla}\left(\frac{\gamma(\rho)}{\rho}\right)\times\boldsymbol{\nabla}\,\Sigma\ , \tag{3.30}$$

with no any contribution from the diffusive term. The irrotational case is recovered in linear drift $\gamma(\rho)=\rho$.

The final expression of the NSE (3.13) is given by

$$i\hbar\,\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2\,m}\,\Delta\,\psi+\Big[W(\rho,\,\Sigma)+i\,\mathcal{W}(\rho,\,\Sigma)\Big]\,\psi+V(\boldsymbol{x})\,\psi\ , \tag{3.31}$$

with the nonlinearities

$$W(\rho,\,\Sigma)=\frac{m}{2}\left(\frac{\partial\,\gamma(\rho)}{\partial\,\rho}-1\right)\left(\frac{\boldsymbol{j}_0}{\rho}\right)^2+m\,D\,f(\rho)\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}_0}{\rho}\right)+G(\rho)\ , \tag{3.32}$$

and

$$\mathcal{W}(\rho,\,\Sigma)=-\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\left\{[\gamma(\rho)-\rho]\left(\frac{\boldsymbol{j}_0}{\rho}\right)\right\}+\frac{\hbar\,D}{2\,\rho}\,\boldsymbol{\nabla}\cdot[f(\rho)\,\boldsymbol{\nabla}\,\rho]\ . \tag{3.33}$$

Eqs. (3.32)-(3.33) differ from the one obtained in Ref. [25] where a family of NSE was derived in the stochastic quantization framework starting from the most general nonlinear classical kinetics compatible with constant diffusion coefficient $D=\hbar/2\,m$. In particular, the real nonlinearity W arising in the stochastic quantization is found to depend only on field ρ, in contrast with expression (3.32), where functional W depends on both fields ρ and Σ.

Remarkably, we observe that when the kinetics of the system is governed by a linear drift, with $\gamma(\rho)=\rho$, the expression of nonlinear terms (3.32) and (3.33) simplify to

$$W(\rho,\,\Sigma)=m\,D\,\widetilde{f}(\rho)\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}_0}{\rho}\right)+G(\rho)\ , \tag{3.34}$$

and

$$\mathcal{W}(\rho,\,\Sigma)=\frac{\hbar\,D}{2\,\rho}\,\boldsymbol{\nabla}\cdot\Big[\widetilde{f}(\rho)\,\ln\kappa(\rho)\,\boldsymbol{\nabla}\,\rho\Big]\ , \tag{3.35}$$

where $\widetilde{f}(\rho)=\rho\,(\partial/\partial\,\rho)\,\ln\kappa(\rho)$.

They are determined only through functional $\kappa(\rho)$ which also defines the entropy (2.1) of the ancestor classical system.

IV. EHRENFEST RELATIONS AND CONSERVED QUANTITIES

In this section we study the time evolution of the most important physical observables of the system described by the Hamiltonian density (3.16) with the nonlinear potential

(3.23): mass center, linear and angular momentum and total energy. The proofs are given in Appendix A.

Let us recall that, given an Hermitian operator $\mathcal{O}=\mathcal{O}^\dagger$ associated with a physical observable, its time evolution is given by

$$\frac{d}{d\,t}\langle\mathcal{O}\,\rangle=\frac{i}{\hbar}\int\left(\frac{\delta\,H}{\delta\,\psi}\,\mathcal{O}\,\psi-\psi^*\,\mathcal{O}\,\frac{\delta\,H}{\delta\,\psi^*}\right)\,d\boldsymbol{x}+\left\langle\frac{\partial\,\mathcal{O}}{\partial\,t}\right\rangle\,, \tag{4.1}$$

where the mean value $\langle\mathcal{O}\,\rangle=\int\psi^*\mathcal{O}\,\psi\,d\boldsymbol{x}$. The last term in Eq. (4.1) takes into account a possible explicit time dependence on the operator $\mathcal{O}$.

Observing that the NSE (3.31) can be written in

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}=\mathsf{H}\,\psi\,, \tag{4.2}$$

where

$$\mathsf{H}=-\frac{\hbar^2}{2\,m}\,\Delta+W(\rho,\,\Sigma)+i\,\mathcal{W}(\rho,\,\Sigma)+V(\boldsymbol{x})\,, \tag{4.3}$$

Eq. (4.1) assumes the equivalent expression

$$\frac{d}{d\,t}\langle\mathcal{O}\,\rangle=\frac{i}{\hbar}\left\langle\Big[\mathrm{Re}\,\mathsf{H},\,\mathcal{O}\Big]\right\rangle+\frac{1}{\hbar}\left\langle\Big\{\mathrm{Im}\,\mathsf{H},\,\mathcal{O}\Big\}\right\rangle+\left\langle\frac{\partial\,\mathcal{O}}{\partial\,t}\right\rangle\,, \tag{4.4}$$

where $[\cdot,\,\cdot]$ and $\{\cdot,\,\cdot\}$ indicate the commutator and the anticommutator, respectively.

By choosing $\mathcal{O}=\boldsymbol{x}$, from Eq. (4.1) we obtain the first Ehrenfest relation for the time evolution of the mass center of the system

$$\boldsymbol{v}_{\rm mc}\equiv\frac{d}{d\,t}\,\langle\boldsymbol{x}\rangle=\left\langle\frac{\gamma(\rho)}{\rho}\,\widehat{\boldsymbol{u}}_{\rm drift}\right\rangle\,. \tag{4.5}$$

We observe that only drift nonlinearity appears in this equation whereas the diffusion term makes no contribution. Eq. (4.5) states that, quite generally, $\boldsymbol{v}_{\rm mc}$ is not a motion constant. This fact implies that the quantum system is not Galilei invariant. The origin of the nonconservation of $\boldsymbol{v}_{\rm cm}$ can be found in the difference between quantity $\boldsymbol{p}_{\rm mc}=m\,\boldsymbol{v}_{\rm mc}$ and the expectation value of the momentum operator $\boldsymbol{p}\equiv\langle-i\,\hbar\,\boldsymbol{\nabla}\rangle=\int\rho\,\boldsymbol{\nabla}\,\Sigma\,d\boldsymbol{x}$. These two quantities are equivalent only in the linear drift case. Differently from the former, the latter is in all cases conserved during the time evolution of the system, in absence of the external potential. This can be shown by means of the second Ehrenfest relation which follows from Eq. (4.1) by posing $\mathcal{O}=-i\,\hbar\,\boldsymbol{\nabla}$

$$\frac{d}{d\,t}\,\langle\boldsymbol{p}\rangle=\Big\langle\boldsymbol{F}_{\rm ext}(\boldsymbol{x})\Big\rangle\,. \tag{4.6}$$

The time evolution of the expectation value of momentum is governed only by external potential $V(\boldsymbol{x})$. On the average, the KIP introduce no effect on the dynamics of the system. This is a consequence of the invariance of nonlinearity $W[\rho,\,\Sigma]+i\,\mathcal{W}[\rho,\,\Sigma]$ under uniform space translation.

In the same way, accounting for the invariance of nonlinearity for uniform rotations, the third Ehrenfest relation follows

$$\frac{d}{d\,t}\,\langle\boldsymbol{L}\rangle=\Big\langle\boldsymbol{M}_{\rm ext}(\boldsymbol{x})\Big\rangle\ , \tag{4.7}$$

where $\boldsymbol{M}_{\rm ext}(\boldsymbol{x})=\boldsymbol{x}\times\boldsymbol{F}_{\rm ext}(\boldsymbol{x})$ is the momentum of the external force field. Eq. (4.7) is obtained from Eq. (4.1) after posing $\mathcal{O}=\boldsymbol{x}\times(-i\,\hbar\,\boldsymbol{\nabla})$. Again, the nonlinear terms introduced by KIP as well as nonlinearity $G(\rho)$ make no contribution, on the average, to angular momentum.

Finally, the last relation concerns the total energy of the system given by the Hamiltonian $E\equiv H$. By posing

$$\mathcal{O}=-\frac{\hbar^2}{2\,m}\,\Delta+\frac{1}{\rho}\,U(\rho,\,\Sigma)+V(\boldsymbol{x})\ , \tag{4.8}$$

we have $\langle\mathcal{O}\rangle\equiv E$ and from Eq. (4.1) we obtain

$$\frac{d\,E}{d\,t}=0\ . \tag{4.9}$$

In conclusion, for a constant diffusion process we have shown that in absence of the external potential the system admits three constants of motion: total linear momentum $\langle\boldsymbol{p}\rangle$, total angular momentum $\langle\boldsymbol{L}\rangle$ and total energy E. Such conserved quantities, according to the Noether theorem, follow as a consequence of the invariance of the system under uniform space-time translation and uniform rotation. Moreover, the system is also invariant for global U(1) transformation which implies conservation of the normalization of field ψ throughout the evolution of the system.

In Appendix B we briefly discuss the case of a quantum system with a diffusion coefficient $D(t,\,\boldsymbol{x})$ that depends on time and position. This space-time dependence destroys the invariance of the system under uniform space-time translation and space rotation. As a consequence, all quantities $\langle\mathbf{p}\rangle$, $\langle\mathbf{L}\rangle$ and E are no longer conserved, even for a vanishing external potential.

It should be remarked that the results discussed here, although very general in that they are independent of the form of nonlinearities W and $\mathcal{W}$, are valid only for the class of the

canonical systems. In literature there are many noncanonical NSEs, obtained starting from certain physically motivated conditions, which are worthy of being taken into account. For these equations, the expression of $\mathtt{H}$ appearing on the right hand side of the Schrödinger equation cannot be obtained from Eqs. (3.9) and (3.10) by means of a Hamiltonian function $H = \int \mathcal{H}\, d\boldsymbol{x}$.

Despite this, even for these noncanonical systems the time evolution of the mean values of the quantum operators associated with the observables can be derived through Eq. (4.4), but what is important is that these operators can assume a different definition with respect to the one given in the canonical theory. For instance, in the canonical framework the energy is supplied by the Hamiltonian H of the system, whereas in a noncanonical theory it is identified with the operator $i\,\hbar\,\partial/\partial\,t \equiv \mathtt{H}$. (We remark that in the canonical framework H and $\mathtt{H}$ are, in general, different quantities). Moreover, for a noncanonical theory, conservation of the energy and the momentum do not follow merely from the principle of invariance of the system under space-time translation. Their time evolution depends on the expression of the nonlinearities appearing in the Schrödinger equation. All of this clearly causes a profound difference in the resulting Ehrenfest relations.

For instance, in Ref. [20] a noncanonical Schrödinger equation with complex nonlinearity was derived starting from a Fokker-Planck equation for density field ρ by assuming some physically justified separability conditions. The resulting evolution equation has the real and the imaginary nonlinearity given by $W(\rho,\,\Sigma) = \gamma\,(\Sigma - \langle\Sigma\rangle)$ and $\mathcal{W}(\rho,\,\Sigma) = (\hbar\,D/2)\,\Delta\rho/\rho$, respectively, where γ is a constant related to diffusion coefficient D and such that $D \to 0$ if $\gamma \to 0$. It is easy to see that such nonlinearities cannot be obtained starting from a nonlinear potential $U(\rho,\,\Sigma)$ through Eqs. (3.17) and (3.18). The system described by this NSE turns out to be dumped and dissipative, even in presence of a constant diffusive process. In fact, it can be shown that, following Ref. [20], from Eq. (4.4) it follows $d\,\langle\boldsymbol{p}\rangle/d\,t = \langle\boldsymbol{F}_{\rm ext}\rangle - \gamma\,\langle\boldsymbol{p}\rangle$ and $d\,E/d\,t \equiv d\,\langle\mathtt{H}\rangle/d\,t = -(\gamma/m)\,\langle\boldsymbol{p}^2\rangle$, which is a very different situation with respect to the one discussed in the present paper, with the exception of the trivial case $\gamma = 0$.

V. GAUGE EQUIVALENCE

We introduce a nonlinear gauge transformation of the third kind [30]

$$\psi \to \phi = \psi \, \exp\left(-\frac{i}{\hbar}\, m\, D\, \ln\kappa(\rho)\right) \ , \tag{5.1}$$

which, being a unitary transformation, does not change the amplitude of wave function $|\psi|^2 = |\phi|^2 = \rho$, and transforms the phase Σ of the old field ψ, into phase σ of the new field ϕ according to the equation

$$\sigma = \Sigma - m\, D\, \ln\kappa(\rho) \ . \tag{5.2}$$

Consequently, the nonlinear current (3.26) takes the expression

$$\boldsymbol{j} \to \widetilde{\boldsymbol{j}} = \frac{\boldsymbol{\nabla}\sigma}{m}\,\gamma(\rho) \ . \tag{5.3}$$

with only a nonlinear drift term.

Let us observe that, at the classical level, the similar transformation

$$\boldsymbol{u}'_{\rm drift} = \boldsymbol{u}_{\rm drift} - D\,\boldsymbol{\nabla}\, \ln\kappa(\rho) \ , \tag{5.4}$$

changes total current $\boldsymbol{J} \to \boldsymbol{J}' = \boldsymbol{u}'_{\rm drift}\,\gamma(\rho)$ into another one consisting only of a nonlinear drift term.

Performing the transformation (5.1), Eq. (3.31) becomes

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t} = -\frac{\hbar^2}{2\,m}\,\Delta\,\phi + \left[\widetilde{W}(\rho,\,\sigma) + i\,\widetilde{\mathcal{W}}(\rho,\,\sigma)\right]\,\phi + V(\boldsymbol{x})\,\phi \ , \tag{5.5}$$

where the new nonlinearities $\widetilde{W}(\rho,\,\sigma)$ and $\widetilde{\mathcal{W}}(\rho,\,\sigma)$ are given by

$$\widetilde{W}(\rho,\,\sigma) = \frac{m}{2}\left(\frac{\partial\,\gamma(\rho)}{\partial\,\rho} - 1\right)\left(\frac{\widetilde{\boldsymbol{j}}_0}{\rho}\right)^2 + m\,D^2\left[f_1(\rho)\,\Delta\,\rho + f_2(\rho)\,\left(\boldsymbol{\nabla}\,\rho\right)^2\right] + G(\rho) \ , \tag{5.6}$$

with $\widetilde{\boldsymbol{j}}_0 = \rho\,\boldsymbol{\nabla}\,\sigma/m$,

$$f_1(\rho) = \gamma(\rho)\,\left[\frac{\partial}{\partial\,\rho}\,\ln\kappa(\rho)\right]^2 \ , \tag{5.7}$$

$$f_2(\rho) = \frac{1}{2}\,\frac{\partial\,f_1(\rho)}{\partial\,\rho} \ , \tag{5.8}$$

and

$$\widetilde{\mathcal{W}}(\rho,\,\Sigma) = -\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\left\{\left[\gamma(\rho) - \rho\right]\left(\frac{\widetilde{\boldsymbol{j}}_0}{\rho}\right)\right\} \ . \tag{5.9}$$

Eq. (5.5) is still a NSE with a complex nonlinearity due to the presence of the nonlinear drift term in the quantum current expression (5.3).

Basically, both equations (3.31) and (5.5) are different NSEs describing the same physical system. This is a consequence of the unitary structure of the transformation (5.1) which implies that the probability position density for field ψ and field ϕ assumes the same value at any instant of time [23].

In the case of $\gamma(\rho) = \rho$ expressions (5.6) and (5.9) can be simplified and the NSE (5.5) assumes the form

$$i\hbar\,\frac{\partial\phi}{\partial t} = -\,\frac{\hbar^2}{2\,m}\,\Delta\,\phi + m\,D^2\left[\widetilde{f}_1(\rho)\,\Delta\,\rho + \widetilde{f}_2(\rho)\,\left(\boldsymbol{\nabla}\,\rho\right)^2\right]\phi + G(\rho)\,\phi + V(\boldsymbol{x})\,\phi\ , \tag{5.10}$$

with

$$\widetilde{f}_1(\rho) = \rho\,\left[\frac{\partial}{\partial\,\rho}\,\ln\kappa(\rho)\right]^2\ , \tag{5.11}$$

$$\widetilde{f}_2(\rho) = \frac{1}{2}\,\frac{\partial\,\widetilde{f}_1(\rho)}{\partial\,\rho}\ , \tag{5.12}$$

which contains a purely real nonlinearity depending only on field ρ.

We observe that although Eq. (5.1) transforms the nonlinear current into another one without the diffusive term, NSEs (5.5) and (5.10) contain a dependence from on diffusion coefficient D.

The NSE (5.5) is still canonical. It can be obtained from the following Hamiltonian density

$$\mathcal{H}(\rho,\,\sigma) = \frac{(\boldsymbol{\nabla}\,\sigma)^2}{2\,m}\,\rho + \frac{\hbar^2}{8\,m}\,\frac{(\boldsymbol{\nabla}\,\rho)^2}{\rho} + \widehat{U}(\rho,\,\sigma) + V(\boldsymbol{x})\,\rho\ , \tag{5.13}$$

with nonlinear potential

$$\widehat{U}(\rho,\,\sigma) = \frac{(\boldsymbol{\nabla}\,\sigma)^2}{2\,m}\,\Big[\gamma(\rho) - \rho\Big] - \frac{m\,D^2}{2}\,f_1(\rho)\,\left(\boldsymbol{\nabla}\,\rho\right)^2 + \widetilde{U}(\rho)\ . \tag{5.14}$$

In this sense Eq. (5.1) defines a canonical transformation.

In conclusion, let us make the following observation. Eq. (5.5) admits the following continuity equation

$$\frac{\partial\,\rho}{\partial\,t} + \boldsymbol{\nabla}\cdot\left[\frac{\boldsymbol{\nabla}\sigma}{m}\,\gamma(\rho)\right] = 0\ . \tag{5.15}$$

A natural question is: what kind of NSE is obtained if we quantize a classical system obeying the continuity equation $\partial\,\rho/\partial\,t + \boldsymbol{\nabla}\cdot\boldsymbol{J}' = 0$ with the method described above?

We easily have

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t} = -\;\frac{\hbar^2}{2\,m}\,\Delta\,\phi - \frac{m}{2}\left(\frac{\partial\,\gamma(\rho)}{\partial\,\rho} - 1\right)\left(\frac{\widetilde{\boldsymbol{j}}_0}{\rho}\right)^2\phi$$
$$-\;i\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\left\{\left[\gamma(\rho)-\rho\right]\left(\frac{\widetilde{\boldsymbol{j}}_0}{\rho}\right)\right\}\phi + G(\rho)\,\phi + V(\boldsymbol{x})\,\phi\;, \tag{5.16}$$

where now ρ and σ are independent fields representing the amplitude and phase of wave function ϕ. Eq. (5.16) can be derived through the Hamiltonian density (5.13) with nonlinear potential

$$\widehat{U}_{1}(\rho,\,\sigma) = \frac{(\boldsymbol{\nabla}\,\sigma)^2}{2\,m}\Big[\gamma(\rho)-\rho\Big] + \widetilde{U}(\rho)\;. \tag{5.17}$$

Potentials (5.14) and (5.17) differ for the quantity

$$\overline{U}(\rho) = \widehat{U}(\rho,\,\sigma) - \widehat{U}_{1}(\rho,\,\sigma) = -\frac{m\,D^2}{2}\,f_1(\rho)\,\left(\boldsymbol{\nabla}\,\rho\right)^2\;, \tag{5.18}$$

which depends only on field ρ. This nonlinear potential $\overline{U}(\rho)$ does not affect the continuity equation and thus cannot be obtained starting directly from Eq. (5.15).

VI. SOME EXAMPLES

To illustrate the relevance and applicability of the theory described in the previous sections, we derive and discuss some different NSEs obtained starting from kinetic equations known in literature. In the following Section, for simplicity's sake we omit the arbitrary nonlinear potential $\widetilde{U}(\rho)$ and focus our attention only on the effect yield through the potential introduced by the KIP.

A. Boltzmann-Gibbs-entropy

It is well known that when the many body system is governed by short-range interactions, or when interaction energy is neglecting with respect to the total energy of the system, the suitable entropic functional is given by the BG-entropy

$$S_{\rm BG}(\rho) = -\int\rho\,\ln\left(\rho\right)\,d\boldsymbol{x}\;. \tag{6.1}$$

This entropy arises from Eq. (2.1) by posing $\kappa(\rho) = e\,\rho$ with $a(\rho) = e\,\rho$ and $b(\rho) = 1$. It is readily seen that $\gamma(\rho) = e\,\rho\,c(\rho)$.

Among the many NFPEs compatible with entropy (6.1) we consider the simplest case of linear drift by posing $c(\rho) = 1/e$. Then the continuity equation (3.25) becomes the standard linear Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\left(\boldsymbol{j}_{_0} - D\,\boldsymbol{\nabla}\,\rho\right) = 0\ , \tag{6.2}$$

whereas the evolution equation for the quantum system is given by the following NSE

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t} = -\ \frac{\hbar^2}{2\,m}\,\Delta\,\psi + m\,D\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{j}_{_0}}{\rho}\right)\,\psi + i\,\frac{\hbar}{2}\,D\,\frac{\Delta\,\rho}{\rho}\,\psi + V(\boldsymbol{x})\,\psi\ , \tag{6.3}$$

which is recognized as the canonical sub-family of the class of Doebner-Goldin equations parameterized by diffusion coefficient D. We recall that Eq. (6.2) was obtained in the quantum mechanics theory starting from the study of the physical interpretation of a certain family of diffeomorphismin group [23].

By performing gauge transformation (5.1), Eq. (6.3) becomes

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t} = -\frac{\hbar^2}{2\,m}\,\Delta\,\phi + m\,D^2\,\left[\frac{\Delta\,\rho}{\rho} - \frac{1}{2}\left(\frac{\boldsymbol{\nabla}\,\rho}{\rho}\right)^2\right]\,\phi + V(\boldsymbol{x})\,\phi\ , \tag{6.4}$$

which was studied previously in [35]. In particular, Eq. (6.4) is equivalent to the following linear Schrödinger equation

$$i\,\hbar\!\!\!/\,\frac{\partial\,\chi}{\partial\,t} = -\frac{\hbar\!\!\!/^2}{2\,m}\,\Delta\,\chi + V(\boldsymbol{x})\,\chi\ , \tag{6.5}$$

with $\hbar\!\!\!/ = \hbar\,\sqrt{1-(2\,m\,D/\hbar)^2}$ and field χ is related to hydrodynamic fields ρ and σ through the relation $\chi = \rho^{1/2}\,\exp(i\,\sigma/\hbar\!\!\!/)$.

This appear to be an interesting result. By quantizing a classical system described by MB-entropy the standard linear Schrödinger equation was obtained. In this equation the nonlinear terms describing the interaction between the many particles of the quantum system are absent. This is in accordance with the general statement that MB-entropy is suitable for describing systems with no (or negligible) interaction among the particles.

B. Generalized entropies

In presence of long-range interactions or memory effects persistent in time, it has been argued that MB-entropy may not be appropriate in describing such systems. For this reason, many different versions of Eq. (6.1) have been proposed in literature.

Very recently, Ref. [36, 37] introduced a bi-parametric deformation of the logarithmic function

$$\ln_{\{\kappa,r\}}(x) = \frac{x^{r+\kappa} - x^{r-\kappa}}{2\,\kappa} \ , \tag{6.6}$$

which reduces, in the $(\kappa,\, r) \to (0,\, 0)$ limit, to the standard logarithm: $\ln_{\{0,0\}}(x) = \ln x$. By replacing the logarithmic function in Eq. (6.1) with its generalized version (6.6), we obtain a bi-parametric family of generalized entropies

$$S_{\{\kappa,r\}}(\rho) = -\int \rho \ \ln_{\{\kappa,r\}}(\rho)\ d\boldsymbol{x} \ , \tag{6.7}$$

introduced, for the first time, in Refs. [38, 39]. Remarkably, this family of entropies includes, as special cases, some generalized entropies, well known in literature, used in the study of systems exhibiting distribution with asymptotic power law behavior. Among them we can cite Tsallis-entropy [32] which follows by posing $r = \pm|\kappa|$

$$S_q(\rho) = \int \frac{\rho^q - \rho}{1-q}\, d\boldsymbol{x} \ , \tag{6.8}$$

with $q = 1 \pm 2\,|\kappa|$ and Kaniadakis-entropy [26], for $r = 0$

$$S_{\{\kappa\}}(\rho) = -\int \frac{\rho^{1+\kappa} - \rho^{1-\kappa}}{2\,\kappa}\, d\boldsymbol{x} \ . \tag{6.9}$$

Both these entropies, as well as other one-parameter deformed entropies, originated from Eq. (6.7) [37], can be employed to describe generalized statistical systems such as, for instance, charge particles in electric and magnetic fields [40], 2d-turbulence in pure-electron plasma [41], Bremsstrahlung [42] and anomalous diffusion of the correlated and Lévy type [43, 44].

In addition to the many applications where Tsallis-entropy has been employed [45], Kaniadakis-entropy (6.9) has been successfully applied in the description of the energy distribution of fluxes of cosmic rays [26], whereas the entropy in (6.7) with $\kappa^2 = (r+1)^2 - 1$ has been applied in the generalized statistical mechanical study of q-deformed oscillators in the frame-work of quantum-groups [46].

Despite the topics recalled above, there is currently great interest in studying quantum systems with long-range microscopic interactions. Systems such as quantum wires, which are now possible in practice thanks to recent technological advances, require on the theoretical ground, the development of a quantum (nonlinear) theory capable of capturing the emergent facts [47].

The entropy in (6.7) arises from Eq. (2.1) by posing

$$\ln \kappa(\rho) = \lambda \ \ln_{\{\kappa,r\}}\left(\frac{\rho}{\alpha}\right) \ , \tag{6.10}$$

with $\lambda = (1+r-\kappa)^{(r+\kappa)/2\,\kappa}/(1+r+\kappa)^{(r-\kappa)/2\,\kappa}$ and $\alpha = [(1+r-\kappa)/(1+r+\kappa)]^{1/2\,\kappa}$.
Among the many different possibilities, we discuss the case of linear drift with $\gamma(\rho) = \rho$. By taking into account Eq. (6.10) we have continuity equation (3.25) with

$$f(\rho) = a_+\rho^{r+\kappa} - a_-\rho^{r-\kappa} \ , \tag{6.11}$$

where $a_\pm = (r \pm \kappa)\,(1 + r \pm \kappa)/2\,\kappa$ are constants.
The associated NSE assumes the expression

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t} = -\ \frac{\hbar^2}{2\,m}\Delta\,\phi + m\,D^2\,\frac{f(\rho)}{\rho}\ \left[f(\rho)\,\Delta\,\rho + \widetilde{f}(\rho)\ (\boldsymbol{\nabla}\,\rho)^2\right]\phi + V(\boldsymbol{x})\,\phi \ , \tag{6.12}$$

with

$$\widetilde{f}(\rho) = b_+\,\rho^{r+\kappa-1} - b_-\,\rho^{r-\kappa-1} \ , \tag{6.13}$$

and $b_\pm = a_\pm\,(r \pm \kappa - 1/2)$.
Eq. (6.12) contains only a purely real nonlinearity and reduces to Eq. (6.4) in the $(\kappa, r) \to (0,\,0)$ limit, as well as Eq. (6.7), which reduces to the standard BG-entropy.
In particular, for Tsallis-entropy, the continuity equation (3.25), with

$$f(\rho) = q\,\rho^{q-1} \ , \tag{6.14}$$

becomes the diffusive NFPE [48] while the corresponding NSE is given through Eq. (6.12) with

$$\widetilde{f}(\rho) = \left(q - \frac{3}{2}\right)\,\rho^{q-2} \ , \tag{6.15}$$

and reduces to Eq. (6.4) in the $q \to 1$ limit just as entropy (6.8) reduces to BG-entropy.
We observe that in Refs. [49, 50] the quantization of a classical system described by Tsallis-entropy has been already discussed. There, a NLS compatible with the continuity equation $\partial\,\rho^\mu/\partial\,t + \boldsymbol{\nabla}\cdot(\rho^\mu\,\widehat{\boldsymbol{u}}_{\rm drift}) = 0$ was obtained with a different approach. The nonlinearity appearing in the NLS of Refs. [49, 50] reduces, for $\mu = 1$ and $q \to 2 - q$, to the same one reported here.
On the other hand, for Kaniadakis-entropy, the continuity equation is given in Eq. (3.25) with

$$f(\rho) = \frac{1}{2}\,\left[(\kappa+1)\,\rho^{\kappa} - (\kappa-1)\,\rho^{-\kappa}\right] \ , \tag{6.16}$$

which coincides with that proposed in Ref. [26] while the associated NSE is given in Eq. (6.12) with

$$\widetilde{f}(\rho) = \frac{1}{2\,\rho}\left[(\kappa+1)\left(\kappa-\frac{1}{2}\right)\rho^{\kappa} + (\kappa-1)\left(\kappa+\frac{1}{2}\right)\rho^{-\kappa}\right] \ , \tag{6.17}$$

and reduces to Eq. (6.4) in the $\kappa \to 0$ limit just as entropy (6.9) reduces to BG-entropy.

C. Interpolating bosons-fermions-entropy

In Ref. [33], on the basis of the generalized exclusion-inclusion principle the authors introduced a family of NFPEs describing the evolution of a classical system of particles whose statistical behavior interpolates between bosonic and fermionic particles. The equilibrium distribution governed by the EIP can be obtained by maximizing the following entropy

$$S_{\rm EIP}(\rho) = -\int\left[\rho\,\ln\rho - \frac{1}{\kappa}\,(1+\kappa\,\rho)\,\ln(1+\kappa\,\rho)\right]d\boldsymbol{x} \ , \tag{6.18}$$

with $-1 \leq \kappa \leq 1$. In particular, for $\kappa = \pm 1$ we recognize the well-known Bose-Einstein and Fermi-Dirac entropies, whereas intermediary behavior follows for $-1 < \kappa < 1$. Entropy (6.18) can be obtained from Eq. (2.1) by posing $a(\rho) = \rho$ and $b(\rho) = 1 + \kappa\,\rho$.

Some examples of real physical systems where EIP can be usefully applied are to be found in the Bose-Einsten condensation. Typically, the cubic NSE is used to describe the behavior of the condensate by simulating in this way the statistical attraction between the many bodies constituting the system. In spite of the simplest cubic interaction, other interactions like the one introduced by the EIP can be adopted to simulate an attraction among the particles.

In the opposite direction, almost-fermionic systems can be found in the study of the motion of electrons and holes in a semiconductor. In fact, while if separately considered electrons and holes are fermions, together they constitute an excited state behaving differently from a fermion or a boson. The same argument can be applied to the Cooper-pair in the superconductivity theory. Such excitation differs from a pure boson state because of the spatial delocalization of the two electrons, which are not completely overlying. Deviation from Bose statistics must be taken into account.

In the following we discuss separately two different choices for functional $\gamma(\rho)$.

In the linear drift case, with $c(\rho) = 1/(1+\kappa\,\rho)$, the evolution equation for field ρ assumes

the expression

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\left(\boldsymbol{j}_{_0}-D\,\frac{\boldsymbol{\nabla}\,\rho}{1+\kappa\,\rho}\right)=0\ , \tag{6.19}$$

which was proposed in Ref. [22]. By means of Eq. (5.1), nonlinear current $\boldsymbol{j}_{_0}-D\,\boldsymbol{\nabla}\,\rho/(1+\kappa\,\rho)\to\widetilde{\boldsymbol{j}}_{_0}$ assumes the standard bilinear form and the corresponding NSE follows from Eq. (5.10)

$$i\,\hbar\,\frac{\partial\,\phi}{\partial\,t}=-\ \frac{\hbar^2}{2\,m}\,\Delta\,\phi+\frac{m\,D^2}{(1+\kappa\,\rho)^2}\left[\frac{\Delta\,\rho}{\rho}-\frac{1-3\,\kappa\,\rho}{2\,(1+\kappa\,\rho)}\left(\frac{\boldsymbol{\nabla}\,\rho}{\rho}\right)^2\right]\phi+V(\boldsymbol{x})\,\phi\ . \tag{6.20}$$

We can observe that in Eq. (6.20) the EIP is accounting through a diffusion process and its effect vanishes in the $D\to 0$ limit where it reduces to the standard linear Schrödinger equation. Eq. (6.20) has a purely real nonlinearity depending only on field ρ.

In a different way, by making the choice $c(\rho)=1$, the continuity equation (3.25) becomes

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\Big[\boldsymbol{j}_{_0}\,(1+\kappa\,\rho)-D\,\boldsymbol{\nabla}\,\rho\Big]=0\ . \tag{6.21}$$

The gauge transformation changes nonlinear current $\boldsymbol{j}\equiv\boldsymbol{j}_{_0}\,(1+\kappa\,\rho)-D\,\boldsymbol{\nabla}\,\rho\to\widetilde{\boldsymbol{j}}\equiv\widetilde{\boldsymbol{j}}_{_0}\,(1+\kappa\,\rho)$ containing only a nonlinear drift term and Eq. (6.21) reduces to

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot\Big[\widetilde{\boldsymbol{j}}_{_0}\,(1+\kappa\,\rho)\Big]=0\ . \tag{6.22}$$

This equation was introduced at the classical level in Ref. [33] and subsequently reconsidered at the quantum level in Ref. [24]. The NSE associated with Eq. (6.22) is given by

$$\begin{aligned}i\,\hbar\,\frac{\partial\,\phi}{\partial\,t}=&-\ \frac{\hbar^2}{2\,m}\,\Delta\,\phi+\frac{m\,D^2}{1+\kappa\,\rho}\left[\frac{\Delta\,\rho}{\rho}-\frac{1+2\,\kappa\,\rho}{2\,(1+\kappa\,\rho)}\left(\frac{\boldsymbol{\nabla}\,\rho}{\rho}\right)^2\right]\phi\\&+\ \kappa\,\frac{m}{\rho}\left(\frac{\widetilde{\boldsymbol{j}}}{1+\kappa\,\rho}\right)^2\phi-i\,\kappa\,\frac{\hbar}{2\,\rho}\,\boldsymbol{\nabla}\cdot\left(\frac{\widetilde{\boldsymbol{j}}\,\rho}{1+\kappa\,\rho}\right)\phi+V(\boldsymbol{x})\,\phi\ .\end{aligned} \tag{6.23}$$

We observe that Eq. (6.23) still has a complex nonlinearity due to the nonlinear structure of quantum current $\widetilde{\boldsymbol{j}}$ and both the nonlinearities W and $\mathcal{W}$ depend on fields ρ and σ. Moreover, in Eq. (6.23), EIP is accounted through a nonlinear drift term and survives even in absence of a diffusion process ($D\to 0$).

Factor $(1+\kappa\,\rho)$ in nonlinear current $\widetilde{\boldsymbol{j}}$ takes into account the EIP in the many particle system. In fact, transition probability (2.15) from site $\boldsymbol{x}$ to $\boldsymbol{y}$ is defined as $\pi(t,\,\boldsymbol{x}\to\boldsymbol{y})=r(t,\,\boldsymbol{x},\,\boldsymbol{x}\to\boldsymbol{y})\,\rho(t,\,\boldsymbol{x})\,[1+\kappa\,\rho(t,\,\boldsymbol{y})]$. For $\kappa\neq 0$ the EIP holds and parameter κ quantifies to what extent particle kinetics is affected by the particle population of the arrival site.

If $\kappa > 0$ the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{y})$ contains an inclusion principle. In fact, the population density at arrival point $\boldsymbol{y}$ stimulates the particle transition and therefore transition probability increases linearly with $\rho(t,\, \boldsymbol{y})$. Where $\kappa < 0$ the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{y})$ takes into account the Pauli exclusion principle. If the arrival point $\boldsymbol{y}$ is empty $\rho(t,\, \boldsymbol{y}) = 0$, the $\pi(t,\, \boldsymbol{x} \to \boldsymbol{y})$ depends only on the population of the starting point. If arrival site is populated $0 < \rho(t,\, \boldsymbol{y}) \leq \rho_{max}$, the transition is inhibited. The range of values that parameter κ can assume is limited by the condition that $\pi(t,\, \boldsymbol{x} \to \boldsymbol{y})$ be real and positive as $r(t,\, \boldsymbol{x},\, \boldsymbol{x} \to \boldsymbol{y})$. We may conclude that $\kappa \geq -1/\rho_{max}$.

A physical meaning of parameter κ can be supplied by the following considerations. We recall that Bose-Einstein and Fermi-Dirac statistics originate from the fundamental principle of indistinguishability in quantum mechanics which is closely related to the symmetrization of the wave function. Completely symmetric wave functions are used to describe bosons while fermions are described by completely anti-symmetric wave functions. Thus, intermediate statistics arise in presence of incomplete symmetrization or anti-symmetrization of the wave function and the concept of degree of symmetrization or degree of anti-symmetrization has been introduced [33]. Parameter κ has the meaning of degree of indistinguishability of fermions or bosons, corresponding to the degree of symmetrization or anti-symmetrization, respectively. Value $\kappa = -1$ corresponds to the case of fermions and denotes a complete anti-symmetric wave function whereas value $\kappa = 1$ corresponds to the case of bosons and denotes a complete symmetric wave function. In addition, value $\kappa = 0$ is associated with classical MB statistics and all the intermediate cases arise when κ assumes all the values between -1 and 1.

Eq. (6.23), for $D = 0$, was obtained previously in Ref. [24], where the canonical quantization of the classical system obeying EIP was accounted for. As discussed in Section V, Eq. (6.23) differs from the NSE obtained in [24] for a real nonlinearity originated from nonlinear potential $\widetilde{U}(\rho) = -m\, D^2\, (\boldsymbol{\nabla}\, \rho)^2/\rho\, (1 + \kappa\rho)$ and depending only on field ρ.
Finally, we observe that different from Eq. (6.20), Eq. (6.23) has vorticity different from zero. The Clebsh potentials corresponding to current $\widetilde{\boldsymbol{j}} = (\boldsymbol{\nabla}\sigma/m)\, \rho\, (1 + \kappa\, \rho)$ are given by $\nu = 1 + \kappa\, \rho$, $\lambda = \sigma$ and $\mu = const$ and vorticity assumes the expression

$$\boldsymbol{\omega} = \frac{\kappa}{m}\, \boldsymbol{\nabla}\, \rho \times \boldsymbol{\nabla}\, \sigma\ . \tag{6.24}$$

In Ref. [51, 52] localized, static, fermion-like vortex solutions ($\kappa < 0$) were obtained and

studied starting from Eq. (6.23) with $D = 0$. We observe that in [51, 52] a different definition of the Clebsh potentials corresponding to $\mu = \lambda = \sigma$ and $\nu = \kappa\,\rho$ was adopted. Despite this, vorticity assumes the same expression that is given by Eq. (6.24) in both cases.
EIP vortex solutions are important on the theoretical ground and for interpretation of experimental results of several applications. For instance, they can be employed in the study of fermion-like vortices observed in ^{3}He-A superfluidity or in heavy fermion superconductors UPt_3 and $U_{0.97}Th_{0.03}Be_{13}$ [53, 54, 55].

VII. CONCLUSIONS

We have presented the quantization of a classical system of interacting particles obeying a kinetic interaction principle. The KIP both fixes the expression of the Fokker-Planck equation describing the kinetic evolution of the system and imposes the form of its entropy. In the framework of canonical quantization, we have introduced a class of NSEs with complex nonlinearity obtained from the classical system obeying KIP. The form of nonlinearity $\Lambda(\psi^*,\,\psi)$ is determined by functional $\kappa(\rho)$, which also fixes the form of the entropy of ancestor classical system.

Among the many interesting solutions of the family of NSEs (3.31) we observe that for a free system with $V(\boldsymbol{x}) = 0$, and posing $G(\rho) = 0$, the planar wave

$$\psi(t,\,\boldsymbol{x}) = A\,\exp\left(-\frac{i}{\hbar}\,(\omega\,t - \boldsymbol{k}\cdot\boldsymbol{x})\right)\ , \tag{7.1}$$

with constant amplitude $A = const$ is the simplest solution, where the relationship between ω and $\boldsymbol{k}$ is given by

$$\omega = \frac{\hbar^2\,\boldsymbol{k}^2}{2\,m}\,\frac{\partial\,\gamma(\rho)}{\partial\,\rho}\bigg|_{\rho=A^2}\ , \tag{7.2}$$

and reduces to the standard dispersion relation for $\gamma(\rho) = \rho$.

When the quantum system is in a stationary state such that $\partial\,\rho_{\rm s}/\partial\,t = 0$, the relationships between distribution $\rho_{\rm s}$ and phase $\Sigma_{\rm s}$ follow from Eq. (3.25)

$$\rho_{\rm s} = \kappa^{-1}\left(\exp\left(\frac{\Sigma_{\rm s}(\boldsymbol{x})}{m\,D} - \beta'\right)\right)\ , \tag{7.3}$$

which mimics the classical equilibrium distribution (2.12), as can be seen by replacing $\Sigma_{\rm s}(\boldsymbol{x})/m\,D$ with $-\beta\,\mathcal{E}(\boldsymbol{x})$. Despite this, we stress that such an analogy is purely formal. The equivalence between Eqs. (2.12) and (7.3) requires that the following relation

$\Sigma_{\rm s}(\boldsymbol{x})/m\,D = -\beta\,\mathcal{E}(\boldsymbol{x})$ must hold. In the general case the expression of stationary phase $\Sigma_{\rm s}(\boldsymbol{x})$ must be obtained from Eq. (3.24), after posing $\partial\,\Sigma_{\rm s}/\partial\,t = 0$, with ρ given through Eq. (7.3).

Finally, another interesting class of possible solutions are solitons. It is well know that soliton solutions in NSE arise when the dispersive effects, principally due to term $-(\hbar^2/2\,m)\,\Delta\,\psi$, is exactly balanced by the nonlinear terms. The existence of this class of solutions depends on the particular form of functionals $\gamma(\rho)$ and $\kappa(\rho)$ which fix the expression of nonlinearities $W(\rho,\,\Sigma)$ and $\mathcal{W}(\rho,\,\Sigma)$. A special situation, where soliton solutions are found within the NSEs derived in this paper, is given by the EIP-equation (6.23) with $D = 0$ [24] where $\gamma(\rho) = \rho\,(1+\kappa\,\rho)$ and $\kappa(\rho) = \rho/(1+\kappa\,\rho)$.
The study of soliton solutions for other functional choices of $\gamma(\rho)$ and $\kappa(\rho)$, like, for instance, the ones related to the generalized entropies discussed in Section VI-B, is a very important task which deserves further research. These solutions may lead to practical applications. In fact, in recent years there has been great interest in the formulation of models where solitons can interact with a long-range force [56]. Typical nonlinear models supporting solitons, like the sine-Gordon model, arise from short-range forces. However, there is experimental evidence that most real transfer mechanisms have long-range interaction, as noted in condensed matter theory [57] or in spin glasses [58].

APPENDIX A.

We present proof of the Ehrenfest equations discussed in Section IV. In the following we assume uniform boundary conditions on the fields in order to neglect the surface terms.

Let us rewrite Eq. (4.1) in a more suitable form. Accounting for the relation

$$\frac{\delta}{\delta\,\psi} = \psi^*\left(\frac{\delta}{\delta\,\rho} - \frac{i\,\hbar}{2\,\rho}\,\frac{\delta}{\delta\,\Sigma}\right)\ , \tag{A.1}$$

Eq. (4.1) becomes

$$\frac{d}{d\,t}\langle\mathcal{O}\,\rangle = \frac{i}{\hbar}\left\langle\,\left[\frac{\delta\,H}{\delta\,\rho},\,\mathcal{O}\right]\,\right\rangle + \frac{1}{2}\left\langle\,\left\{\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma},\,\mathcal{O}\right\}\,\right\rangle + \left\langle\frac{\partial\,\mathcal{O}}{\partial\,t}\right\rangle\,. \tag{A.2}$$

Eq. (4.5) can be obtained starting from Eq. (A.2) by posing $\mathcal{O} = \boldsymbol{x}$

$$\frac{d}{d\,t}\,\langle\boldsymbol{x}\rangle \;=\; \frac{i}{\hbar}\int\left[\psi^*\,\frac{\delta\,H}{\delta\,\rho}\,\boldsymbol{x}\,\psi - \psi^*\,\boldsymbol{x}\,\frac{\delta\,H}{\delta\,\rho}\,\psi\right]\,d\boldsymbol{x} + \frac{1}{2}\int\left[\psi^*\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\boldsymbol{x}\,\psi + \psi^*\,\boldsymbol{x}\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\psi\right]\,d\boldsymbol{x}$$

$$
\begin{aligned}
&= \int \boldsymbol{x}\, \frac{\delta H}{\delta \Sigma}\, d\boldsymbol{x} \\
&= -\int \boldsymbol{x}\, \boldsymbol{\nabla} \cdot \left[\frac{\boldsymbol{\nabla}\Sigma}{m}\, \gamma(\rho) - D\, f(\rho)\, \boldsymbol{\nabla}\, \rho \right] d\boldsymbol{x} \\
&= \int \left[\frac{\boldsymbol{\nabla}\Sigma}{m}\, \gamma(\rho) - D\, f(\rho)\, \boldsymbol{\nabla}\, \rho \right] d\boldsymbol{x} \\
&= \int \frac{\boldsymbol{\nabla}\Sigma}{m}\, \gamma(\rho)\, d\boldsymbol{x} - D \int \boldsymbol{\nabla}\, F(\rho)\, d\boldsymbol{x} \\
&= \left\langle \frac{\gamma(\rho)}{\rho}\, \widehat{\boldsymbol{u}}_{\rm drift} \right\rangle ,
\end{aligned} \tag{A.3}
$$

where

$$
F(\rho) = \int^{\rho} f(\rho')\, d\rho' \; . \tag{A.4}
$$

To show the validity of Eq. (4.6) we pose $\mathcal{O} = -i\,\hbar\,\boldsymbol{\nabla}$ in Eq. (A.2) so that

$$
\begin{aligned}
\frac{d}{d\,t}\, \langle \boldsymbol{p} \rangle \; &= \int \left[\psi^*\, \frac{\delta H}{\delta \rho}\, \boldsymbol{\nabla}\, \psi - \psi^*\, \boldsymbol{\nabla} \left(\frac{\delta H}{\delta \rho}\, \psi \right) \right] d\boldsymbol{x} \\
&- \; i\, \frac{\hbar}{2} \int \left[\psi^*\, \frac{1}{\rho}\, \frac{\delta H}{\delta \Sigma}\, \boldsymbol{\nabla}\, \psi + \psi^*\, \boldsymbol{\nabla} \left(\frac{1}{\rho}\, \frac{\delta H}{\delta \Sigma}\, \psi \right) \right] d\boldsymbol{x} \\
&= \int \frac{\delta H}{\delta \rho}\, \left(\psi^*\, \boldsymbol{\nabla}\, \psi + \psi\, \boldsymbol{\nabla}\, \psi^* \right) - i\, \frac{\hbar}{2} \int \frac{1}{\rho}\, \frac{\delta H}{\delta \Sigma}\, \left(\psi^*\, \boldsymbol{\nabla}\, \psi - \psi\, \boldsymbol{\nabla}\, \psi^* \right)\, d\boldsymbol{x} \\
&= \int \left(\frac{\delta H}{\delta \rho}\, \boldsymbol{\nabla}\, \rho + \frac{\delta H}{\delta \Sigma}\, \boldsymbol{\nabla}\, \Sigma \right) d\boldsymbol{x} \; ,
\end{aligned} \tag{A.5}
$$

where an integration by parts has been performed, and we have posed

$$
\psi^*\, \boldsymbol{\nabla}\, \psi + \psi\, \boldsymbol{\nabla}\, \psi^* = \boldsymbol{\nabla}\, \rho \; , \tag{A.6}
$$

$$
\psi^*\, \boldsymbol{\nabla}\, \psi - \psi\, \boldsymbol{\nabla}\, \psi^* = i\, \frac{2}{\hbar}\, \rho\, \boldsymbol{\nabla}\, \Sigma \; . \tag{A.7}
$$

Taking into account the relation

$$
\boldsymbol{\nabla}\, \mathcal{H} = \frac{\delta H}{\delta \rho}\, \boldsymbol{\nabla}\, \rho + \frac{\delta H}{\delta \Sigma}\, \boldsymbol{\nabla}\, \Sigma + \rho\, \boldsymbol{\nabla}\, V(\boldsymbol{x}) \; , \tag{A.8}
$$

from Eq. (A.5) it follows

$$
\begin{aligned}
\frac{d}{d\,t}\, \langle \boldsymbol{p} \rangle \; &= \int \boldsymbol{\nabla}\, \mathcal{H}\, d\boldsymbol{x} - \int \rho\, \boldsymbol{\nabla}\, V(\boldsymbol{x})\, d\boldsymbol{x} \\
&= \langle \boldsymbol{F}_{\rm ext}(\boldsymbol{x}) \rangle \; ,
\end{aligned} \tag{A.9}
$$

Eq. (4.7) can easily be obtained following the same steps used in the proof of Eq. (4.6).

Finally, by posing $\mathcal{O} = -(\hbar^2/2\,m)\,\Delta + U(\rho,\,\Sigma)/\rho + V(\boldsymbol{x})$ in Eq. (A.2), where $U(\rho,\,\Sigma)$ is given in Eq. (3.23), we have

$$
\begin{aligned}
\frac{d\,E}{d\,t} &= \frac{i}{\hbar}\int\left\{\psi^*\,\frac{\delta\,H}{\delta\,\rho}\,\left(-\frac{\hbar^2}{2\,m}\,\Delta+\frac{U}{\rho}+V\right)\,\psi-\psi^*\,\left[\left(-\frac{\hbar^2}{2\,m}\,\Delta+\frac{U}{\rho}+V\right)\,\frac{\delta\,H}{\delta\,\rho}\,\psi\right]\right\}\,d\boldsymbol{x}\\
&+ \frac{1}{2}\int\left\{\psi^*\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\left(-\frac{\hbar^2}{2\,m}\,\Delta+\frac{U}{\rho}+V\right)\,\psi+\psi^*\,\left[\left(-\frac{\hbar^2}{2\,m}\,\Delta+\frac{U}{\rho}+V\right)\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\rho}\,\psi\right]\right\}\,d\boldsymbol{x}\\
&+ \int\rho\,\frac{\partial}{\partial\,t}\left(\frac{U}{\rho}+V\right)\,d\boldsymbol{x}\\
&= -\frac{i\,\hbar}{2\,m}\int\left[\psi^*\,\frac{\delta\,H}{\delta\,\rho}\,\Delta\,\psi-\psi^*\,\Delta\left(\frac{\delta\,H}{\delta\,\rho}\,\psi\right)\right]\,d\boldsymbol{x}\\
&+ \frac{i}{\hbar}\int\left[\psi^*\,\frac{\delta\,H}{\delta\,\rho}\,\left(\frac{U}{\rho}+V\right)\,\psi-\psi^*\,\left(\frac{U}{\rho}+V\right)\,\frac{\delta\,H}{\delta\,\rho}\,\psi\right]\,d\boldsymbol{x}\\
&- \frac{\hbar^2}{4\,m}\int\left[\psi^*\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\Delta\,\psi+\psi^*\,\Delta\left(\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\psi\right)\right]\,d\boldsymbol{x}\\
&+ \frac{1}{2}\int\left[\psi^*\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\left(\frac{U}{\rho}+V\right)\,\psi+\psi^*\,\left(\frac{U}{\rho}+V\right)\,\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\psi\right]\,d\boldsymbol{x}+\int\rho\,\frac{\partial}{\partial\,t}\left(\frac{U}{\rho}\right)\,d\boldsymbol{x}\\
&= -\frac{i\,\hbar}{2\,m}\int\frac{\delta\,H}{\delta\,\rho}\,\left(\psi^*\,\Delta\,\psi-\psi\,\Delta\,\psi^*\right)\,d\boldsymbol{x}-\frac{\hbar^2}{4\,m}\int\frac{1}{\rho}\,\frac{\delta\,H}{\delta\,\Sigma}\,\left(\psi^*\,\Delta\,\psi+\psi\,\Delta\,\psi^*\right)\,d\boldsymbol{x}\\
&+ \int\left[\frac{\delta\,H}{\delta\,\Sigma}\,\left(\frac{U}{\rho}+V\right)+\frac{\partial\,U}{\partial\,t}-\frac{U}{\rho}\,\frac{\partial\,\rho}{\partial\,t}\right]\,d\boldsymbol{x}\;,
\end{aligned}
\tag{A.10}
$$

where a double integration by parts has been performed. Taking into account

$$
-\frac{i\,\hbar}{2\,m}\,\left(\psi^*\,\Delta\,\psi-\psi\,\Delta\,\psi^*\right)=\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\,\Sigma}{m}\,\rho\right)\;,
\tag{A.11}
$$

$$
\psi^*\,\Delta\,\psi+\psi\,\Delta\,\psi^*=2\,\rho\,\left[\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}}-\left(\frac{\boldsymbol{\nabla}\,\Sigma}{\hbar}\right)^2\right]\;,
\tag{A.12}
$$

which follow from Eq. (3.2), and the relation

$$
\frac{\partial\,U}{\partial\,t}=\left(\frac{\delta}{\delta\,\rho}\int U\,d\boldsymbol{x}\right)\,\frac{\partial\,\rho}{\partial\,t}+\left(\frac{\delta}{\delta\,\Sigma}\int U\,d\boldsymbol{x}\right)\,\frac{\partial\,\Sigma}{\partial\,t}\;,
\tag{A.13}
$$

Eq. (A.10) becomes

$$
\begin{aligned}
\frac{d\,E}{d\,t} &= \int\left\{\frac{\delta\,H}{\delta\,\rho}\,\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\,\Sigma}{m}\,\rho\right)-\frac{\hbar^2}{2\,m}\,\frac{\delta\,H}{\delta\,\Sigma}\,\left[\frac{\Delta\,\sqrt{\rho}}{\sqrt{\rho}}-\left(\frac{\boldsymbol{\nabla}\,\Sigma}{\hbar}\right)^2\right]\right\}\,d\boldsymbol{x}\\
&+ \int\left[\frac{\delta\,H}{\delta\,\Sigma}\,\left(\frac{U}{\rho}+V\right)+\left(\frac{\delta}{\delta\,\rho}\int U\,d\boldsymbol{x}-\frac{U}{\rho}\right)\,\frac{\partial\,\rho}{\partial\,t}+\left(\frac{\delta}{\delta\,\Sigma}\int U\,d\boldsymbol{x}\right)\,\frac{\partial\,\Sigma}{\partial\,t}\right]\,d\boldsymbol{x}\;.
\end{aligned}
\tag{A.14}
$$

By using the relations

$$\frac{\hbar^2}{2\,m}\left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}-\left(\frac{\boldsymbol{\nabla}\,\Sigma}{\hbar}\right)^2\right]=\frac{\delta}{\delta\,\rho}\int U\,d\boldsymbol{x}-\frac{\delta\,H}{\delta\rho}+V\ , \tag{A.15}$$

$$\boldsymbol{\nabla}\cdot\left(\frac{\boldsymbol{\nabla}\,\Sigma}{m}\,\rho\right)=\frac{\delta}{\delta\,\Sigma}\int U\,d\boldsymbol{x}-\frac{\delta\,H}{\delta\Sigma}\ , \tag{A.16}$$

which follow from Eqs. (3.11), (3.16) and (3.23), and motion equations (3.9) and (3.10), we obtain

$$\begin{aligned}\frac{d\,E}{d\,t}\ &=\ \int\left[\frac{\delta\,H}{\delta\,\rho}\left(\frac{\delta}{\delta\,\Sigma}\int U\,d\boldsymbol{x}-\frac{\delta\,H}{\delta\Sigma}\right)-\frac{\delta\,H}{\delta\,\Sigma}\left(\frac{\delta}{\delta\,\rho}\int U\,d\boldsymbol{x}-\frac{\delta\,H}{\delta\rho}+V\right)\right]d\boldsymbol{x}\\ &+\ \int\left[\frac{\delta\,H}{\delta\,\Sigma}\left(\frac{U}{\rho}+V\right)+\frac{\delta\,H}{\delta\Sigma}\left(\frac{\delta}{\delta\,\rho}\int U\,d\boldsymbol{x}-\frac{U}{\rho}\right)-\frac{\delta\,H}{\delta\rho}\left(\frac{\delta}{\delta\,\Sigma}\int U\,d\boldsymbol{x}\right)\right]d\boldsymbol{x}\\ &=\ 0\ .\end{aligned} \tag{A.17}$$

APPENDIX B.

We briefly discuss the generalization of the theory for quantum systems obeying the KIP and undergoing a diffusive process with a diffusion coefficient $D(t,\,\boldsymbol{x})$ depending both on time and space position.

Given the following Hamiltonian density

$$\mathcal{H}(\rho,\,\Sigma)=\frac{(\boldsymbol{\nabla}\,\Sigma)^2}{2\,m}\,\gamma(\rho)+\frac{\hbar^2}{8\,m}\,\frac{(\boldsymbol{\nabla}\,\rho)^2}{\rho}-D(t,\,\boldsymbol{x})\,\gamma(\rho)\,\boldsymbol{\nabla}\,\ln\kappa(\rho)\cdot\boldsymbol{\nabla}\Sigma+\widetilde{U}(\rho)+V(\boldsymbol{x})\,\rho\ , \tag{B.1}$$

from the Hamilton equations (3.9)-(3.10) we obtain the NSE

$$i\,\hbar\,\frac{\partial\,\psi}{\partial\,t}=-\frac{\hbar^2}{2\,m}\,\Delta\,\psi+\Big[W(\rho,\,\Sigma)\ +i\,\mathcal{W}(\rho,\,\Sigma)\Big]\,\psi+G(\rho)\,\psi+V(\boldsymbol{x})\,\Psi\ , \tag{B.2}$$

with nonlinearities

$$W(\rho,\,\Sigma)=\frac{m}{2}\,\left(\frac{\partial\,\gamma(\rho)}{\partial\,\rho}-1\right)\,\left(\frac{\boldsymbol{j}_0}{\rho}\right)^2+m\,\gamma(\rho)\,\frac{\partial}{\partial\,\rho}\,\ln\kappa(\rho)\,\boldsymbol{\nabla}\cdot\left(D(t,\,\boldsymbol{x})\,\frac{\boldsymbol{j}_0}{\rho}\right)+G(\rho)\ , \tag{B.3}$$

and

$$\mathcal{W}(\rho,\,\Sigma)=-\frac{\hbar}{2\,m\,\rho}\,\boldsymbol{\nabla}\Big\{[\gamma(\rho)-\rho]\,\boldsymbol{\nabla}\,\Sigma\Big\}+\frac{1}{2\,\rho}\,\boldsymbol{\nabla}\cdot[D(t,\,\boldsymbol{x})\,\gamma(\rho)\,\boldsymbol{\nabla}\,\ln\kappa(\rho)]\ . \tag{B.4}$$

The system described by Hamiltonian (B.1) is dissipative since $d\,E/d\,t\neq 0$. This is a consequence of the time dependence of D which breaks the invariance of Eq. (B.1) under

uniform time translation. In the same way, linear momentum as well as angular momentum are no longer conserved, even in absence of the external potential, as a consequence of the position dependence of D which breaks the invariance of Eq. (B.1) under uniform space translation and uniform space rotation. This can also be seen from the Ehrenfest relations

$$\frac{d}{dt}\langle \boldsymbol{x}\rangle = \left\langle \frac{\gamma(\rho)}{\rho}\,\widehat{\boldsymbol{u}}_{\rm drift}\right\rangle - \left\langle D(t,\,\boldsymbol{x})\,f(\rho)\,\boldsymbol{\nabla}\,\ln\rho\right\rangle \;, \tag{B.5}$$

$$\frac{d}{dt}\langle \boldsymbol{p}\rangle = -m\left\langle A(\rho,\,\Sigma)\,\boldsymbol{\nabla}\,D(t,\,\boldsymbol{x})\right\rangle + \left\langle \boldsymbol{F}_{\rm ext}(\boldsymbol{x})\right\rangle \;, \tag{B.6}$$

$$\frac{d}{dt}\langle \boldsymbol{L}\rangle = -m\left\langle A(\rho,\,\Sigma)\left(\boldsymbol{x}\times\boldsymbol{\nabla}\,D(t,\,\boldsymbol{x})\right)\right\rangle + \left\langle \boldsymbol{M}_{\rm ext}(\boldsymbol{x})\right\rangle \;, \tag{B.7}$$

$$\frac{dE}{dt} = -m\left\langle A(\rho,\,\Sigma)\,\frac{\partial}{\partial t}\,D(t,\,\boldsymbol{x})\right\rangle \;, \tag{B.8}$$

where $A(\rho,\,\Sigma) = f(\rho)\,\boldsymbol{\nabla}\,\ln\rho\cdot\,\widehat{\boldsymbol{u}}_{\rm drift}$.

Finally, the gauge transformation described in Section V cannot be performed, in general, when the diffusion coefficient has spatial dependence. In fact, the transformation in (5.1) is well defined only if the following condition is fulfilled

$$\boldsymbol{\nabla}\times\Big[D(t,\,\boldsymbol{x})\,\boldsymbol{\nabla}\,\ln\kappa(\rho)\Big] = 0 \;, \tag{B.9}$$

as can be seen by applying the curl operator to both sides of equation

$$\boldsymbol{\nabla}\,\sigma = \boldsymbol{\nabla}\,\Sigma - m\,D(t,\,\boldsymbol{x})\,\boldsymbol{\nabla}\,\ln\kappa(\rho) \;, \tag{B.10}$$

which follows from Eqs. (3.26) and (5.3). We remark that if the dynamics of the system evolves in one spatial dimension, Eq. (B.9) is trivially verified and the transformation in (5.1) can in all cases be accomplished.

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